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New Level-0 Action of $U_q(\widehat{\mathfrak{sl}}_2)$ on Level-1 Modules*

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ABSTRACT

We define a level-0 action of $U_q(\widehat{\mathfrak{sl}}_2)$ on the sum of level-1 irreducible highest weight modules. With the aid of the affine Hecke algebras, this action is realized on the basis created by the vertex operators. This is a q-analogue of the Yangian symmetry in conformal field theory.

1. Introduction

At the USC meeting, a new symmetry structure in conformal field theory (CFT), the Yangian symmetry of the spinon basis, was introduced[7]. The aim of this paper is to investigate the q-deformation of this mathematical structure. Before stating the main achievement in this paper, let us briefly review the developments in this subject.

Results suggesting a possible fermionic description of the CFT spectrum were

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first obtained by Ref.[26] for certain critical lattice models. Closely related results appeared earlier in the mathematical literature (see e.g., Ref.[28]). Analysis of the spectrum of these models from the Bethe ansatz yielded fermionic expressions for the conformal partition functions (see Refs.[25,30,27,29,18,1,2,34,20] for further developments). This suggested a possible alternative description of the states in CFT, the spinon basis of Refs.[4,6].

The underlying algebraic structure, the Yangian symmetry, has been suggested from quite a different direction. The relevant subject was the diagonalization of the Calogero–Sutherland model with spin degrees of freedom[8,33,3] and the Haldane–Shastry model[22,32]. Their spectrum is described by an asymptotic Bethe ansatz[33]. However the multiplicity of states had never been explained until the works[23,3], which show that the model has a symmetry generated by a Yangian[13]. The multiplicities are identified with the dimensions of the highest weight representations of the Yangian[15,9].

It turns out that the level-1 integrable highest weight module (IHWM) of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ has the same Yangian structure as the space of states of these long range interaction models at a special value of the coupling constant. This fact was established by constructing the new spinon basis in the level-1 IHWM[4,6] and considering the action of the Yangian on it. The spinon basis is generated by the spin-1/2 primary field operator in the level-1 $su(2)$ WZW model. The character formula derived in the spinon basis coincides with the fermionic representation. Recently, the extension of spinon basis to the higher level case has also been investigated[5].

Now let us turn to our motivation in this work. In Refs.[16,17], Faddeev and Takhtajan analysed the structure of the space of the states in the XXX model in the infinite volume limit. In Refs.[12,24], the symmetry structure of the off-critical XXZ model in the infinite volume limit was determined by means of the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The structure of the space of states was understood as the level-0 representation on $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ where $\mathcal{H}$ is the direct sum of the level-1 irreducible highest weight $U_q(\widehat{\mathfrak{sl}}_2)$-modules. The vacuum states are identified with certain degree operators, and the excited states are created by the type II vertex operator acting on the former.

This is similar to the Yangian structure we mentioned above, in the sense that we have a level-0 action. However, the space $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is much bigger than the space $\mathcal{H}$ itself, which goes to the IHWM in the conformal limit. It is still unclear how the level-0 structure on $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is related to the Yangian structure on $\mathcal{H}$ in conformal field theory. As for this point, see the recent paper by Nakayashiki and Yamada[31].

The mathematical content of the Yangian structure was known since Drinfeld and others[14,11]. Namely, there is a functorial construction of Yangian representations from certain representations of the degenerate affine Hecke algebra. In Ref.[10], Chari and Pressley gave a construction of level-0 $U_q(\widehat{\mathfrak{sl}}_n)$-modules from finite-dimensional representations of the affine Hecke algebra. In this paper, we apply their construction to the level-1 $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathcal{H}$ and obtain a new level-0 action on it. For this purpose, following Ref.[6], we use the basis of $\mathcal{H}$ created by the type I vertex operator.

The construction of such a basis was discussed in Ref.[19] in a more general context corresponding to the XYZ model. We do not know if the 'new' level-0 action of $U_q(\widehat{\mathfrak{sl}}_2)$ on level-1 modules is related to some physical models, in a way the Yangian structure is.

Before closing the introduction let us point out some technical details, except for which our construction is a straightforward generalization of the methods given in Refs.[10,4,3,6].

(i) We are forced to consider infinite sums of products of the components of vertex operators. We will introduce suitable completion of the 'N-spinon' space to clarify the mathematical content.

(ii) Because of the fusion relation, which connects the 'N-spinon' sector to the '$(N-2)$-spinon' sector, we must check the compatibility of the level-0 action in these different sectors. This is technically non-trivial.

2. Vertex operators

The purpose of this section is to fix the notation concerning level-1 modules and vertex operators. Throughout this paper we fix a complex number q such that $0 < |q| < 1$.

2.1. The module V_{aff}

Let $U = U_q(\widehat{\mathfrak{sl}}_2)$ denote the quantum affine algebra with the standard generators $e_i, f_i, t_i = q^{h_i}$ $(i = 0, 1)$ and q^d. We shall identify $U_q(\mathfrak{sl}_2)$ with the subalgebra of U generated by e_1, f_1, t_1. We retain the coproduct Δ and the antipode a in Ref.[24], e.g.

$$\begin{aligned}
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,\\
&\Delta(q^h) = q^h \otimes q^h \qquad (h = h_i, d).
\end{aligned}$$

In this paper we shall also use the opposite coproduct $\Delta^{\text{op}} = \sigma \circ \Delta$ $(\sigma x \otimes y = y \otimes x)$. We let U^{op} denote the algbera U equipped with the opposite coalgebra structure determined by Δ^{op}.

Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ be the irreducible two-dimensional module over $U_q(\mathfrak{sl}_2)$. The affinization of V is the U-module

$$V_{\text{aff}} = \text{span}_{\mathbb{C}}\{v_{\varepsilon,n} \mid \varepsilon = \pm, n \in \mathbb{Z}\}$$

given as follows.

$$\begin{aligned}
&e_0 v_{+,n} = v_{-,n+1}, \qquad f_0 v_{-,n} = v_{+,n-1},\\
&e_1 v_{-,n} = v_{+,n}, \qquad f_1 v_{+,n} = v_{-,n},\\
&t_0^{-1} v_{\pm,n} = t_1 v_{\pm,n} = q^{\pm 1} v_{\pm,n}, \qquad q^d v_{\pm,n} = q^n v_{\pm,n}.
\end{aligned}$$

In terms of the generating series

$$v_\varepsilon(z) = \sum_n v_{\varepsilon,n} z^{-n},$$

the action of U reads $e_0 v_+(z) = z v_-(z)$, $e_1 v_-(z) = v_+(z)$, and so on.

2.2. Vertex operators

Let $V(\Lambda_i) = U|i\rangle$ be the integrable level-1 module of U with highest weight vector $|i\rangle$ $(i = 0, 1)$. Set $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$. We have the homogeneous gradation $\mathcal{H} = \bigoplus_{r \geq 0} \mathcal{H}_{-r}$ such that $\mathcal{H}_0 = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$.

For an element $f \in \mathrm{End}_{\mathbb{C}}\mathcal{H}$, the (opposite) adjoint action of $x \in U$ is given by

$$\mathrm{ad}^{\mathrm{op}} x.f = \sum x_{(2)}\ f\ a^{-1}(x_{(1)})$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. For $f, g \in \mathrm{End}_{\mathbb{C}}\mathcal{H}$ and $u \in \mathcal{H}$ we have

$$\begin{aligned} \mathrm{ad}^{\mathrm{op}} x.f \circ g &= \sum \mathrm{ad}^{\mathrm{op}} x_{(2)}.f \circ \mathrm{ad}^{\mathrm{op}} x_{(1)}.g, && (2.1) \\ x.f(u) &= \sum \mathrm{ad}^{\mathrm{op}} x_{(2)}.f \left(x_{(1)} u \right). && (2.2) \end{aligned}$$

With this action we regard $\mathrm{End}_{\mathbb{C}}\mathcal{H}$ as an U^{op}-module.

Proposition 1 *There exists a unique embedding of the U^{op}-module*

$$V_{\mathrm{aff}} \longrightarrow \mathrm{End}_{\mathbb{C}}\mathcal{H}, \qquad v_{\varepsilon,n} \mapsto \tilde{\Phi}^*_{\varepsilon,n} \tag{2.3}$$

such that

$$\tilde{\Phi}^*_{+,0}|0\rangle = |1\rangle, \quad \tilde{\Phi}^*_{-,0}|1\rangle = |0\rangle.$$

The elements $\tilde{\Phi}^*_{\varepsilon,n} \in \mathrm{End}_{\mathbb{C}}\mathcal{H}$ will be referred to as components of the vertex operator of type I. To make contact with the usual formulation, introduce the generating series corresponding to $v_\varepsilon(z)$

$$\tilde{\Phi}^*_\varepsilon(z) = \sum_n \tilde{\Phi}^*_{\varepsilon,n} z^{-n}.$$

Then (2.3) is U^{op}-linear if and only if (i) $\tilde{\Phi}^*_{\varepsilon,n}\mathcal{H}_r \subset \mathcal{H}_{r+n}$ for all n, r, and (ii) the following map is U'-linear:

$$u \otimes v_\varepsilon(z) \mapsto \tilde{\Phi}^*_\varepsilon(z) u. \qquad (u \in \mathcal{H})$$

Here U' signifies the subalgebra generated by e_i, f_i, t_i $(i = 0, 1)$ with the original coproduct Δ. Hence $\tilde{\Phi}^*_\varepsilon(z)$ has the same meaning as the one employed in Ref.[24].

For reasons of weights, $\tilde{\Phi}^*_{\varepsilon,n}$ sends $V(\Lambda_i)$ to $V(\Lambda_{1-i})$. We will sometimes write the restriction to $V(\Lambda_i)$ as

$$\tilde{\Phi}^{*(1-i,i)}_{\varepsilon,n}\ :\ V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}).$$

2.3. Spanning vectors

Set $\mathcal{V}_N = V_{\mathrm{aff}}^{\otimes N}$, $\mathcal{V} = \bigoplus_{N\geq 0}\mathcal{V}_N$. The embedding (2.3) gives rise to an U^{op}-linear map

$$\begin{array}{rcl}\rho \;:\; \mathcal{V} & \longrightarrow & \mathrm{End}_{\mathbb{C}}\mathcal{H}, \\ v_{\varepsilon_1,n_1}\otimes\cdots\otimes v_{\varepsilon_N,n_N} & \mapsto & \tilde{\Phi}^*_{\varepsilon_1,n_1}\cdots\tilde{\Phi}^*_{\varepsilon_N,n_N}.\end{array} \tag{2.4}$$

Acting on the highest weight vector $|0\rangle$ we obtain

$$\rho_0 \;:\; \mathcal{V} \longrightarrow \mathcal{H}, \tag{2.5}$$

$$v_{\varepsilon_1,n_1}\otimes\cdots\otimes v_{\varepsilon_N,n_N} \mapsto \tilde{\Phi}^*_{\varepsilon_1,n_1}\cdots\tilde{\Phi}^*_{\varepsilon_N,n_N}|0\rangle. \tag{2.6}$$

Proposition 2 *The map ρ_0 is $U_q^{\mathrm{op}}(\mathfrak{sl}_2)$-linear and surjective.*

Proof. The $U_q^{\mathrm{op}}(\mathfrak{sl}_2)$-linearity follows from (2.2) and the fact that $|0\rangle$ belongs to the trivial representation of $U_q(\mathfrak{sl}_2)$. Note that the image of ρ_0 contains $|0\rangle, |1\rangle = \tilde{\Phi}^*_{+,0}|0\rangle$, and hence $f_0|0\rangle = \tilde{\Phi}^*_{+,-1}|1\rangle$. Using this and (2.2) one checks readily that the image is also invariant under the action of U. The surjectivity follows from the irreducibility of $V(\Lambda_i)$. □

The vectors (2.6) thus constitute a spanning set of $\mathcal{H}$, and we have an isomorphism of $U_q^{\mathrm{op}}(\mathfrak{sl}_2)$-modules $\mathcal{V}/\mathrm{Ker}\,\rho_0 \xrightarrow{\sim} \mathcal{H}$. In section 3 we will determine $\mathrm{Ker}\,\rho_0$.

2.4. Relations among vertex operators

Let us summarize here the properties of vertex operators which will be used later. To state them we introduce the generating series

$$\varphi_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N) = \frac{\prod_{j=1}^N z_j^{(N-j-p_j+p_N)/2}}{\prod_{j<k}\eta(z_k/z_j)}\tilde{\Phi}^{*(p_0,p_1)}_{\varepsilon_1}(z_1)\cdots\tilde{\Phi}^{*(p_{N-1},p_N)}_{\varepsilon_N}(z_N). \tag{2.7}$$

Here $p_j = 0$ ($j\equiv N \bmod 2$), $= 1$ ($j \not\equiv N \bmod 2$), and we have set

$$\eta(z) = \frac{(q^6z;q^4)_\infty}{(q^4z;q^4)_\infty}, \qquad (z;p)_\infty = \prod_{n=0}^\infty(1-p^nz).$$

Note that (2.7) comprises only integral powers of z_j:

$$\varphi_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N) = \sum_{m_1,\ldots,m_N\in\mathbb{Z}}\varphi_{\varepsilon_1,\cdots,\varepsilon_N;m_1,\cdots,m_N}z_1^{-m_1}\cdots z_N^{-m_N}.$$

Each coefficient $\varphi_{\varepsilon_1,\cdots,\varepsilon_N;m_1,\cdots,m_N}$ has a well-defined action on $\mathcal{H}$.

We use the following R-matrix $\tilde{R}(z) \in \mathrm{End}_{\mathbb{C}} V \otimes V$ regarded as a power series in z.

$$\begin{aligned}
\tilde{R}(z) v_\varepsilon \otimes v_\varepsilon &= \frac{z-q^2}{1-q^2 z} v_\varepsilon \otimes v_\varepsilon, \\
\tilde{R}(z) v_+ \otimes v_- &= \frac{(1-q^2)z}{1-q^2 z} v_+ \otimes v_- + \frac{q(z-1)}{1-q^2 z} v_- \otimes v_+, \\
\tilde{R}(z) v_- \otimes v_+ &= \frac{q(z-1)}{1-q^2 z} v_+ \otimes v_- + \frac{1-q^2}{1-q^2 z} v_- \otimes v_+.
\end{aligned}$$

In general, for an element $A \in \mathrm{End}_{\mathbb{C}} V$ we set

$$\pi_k(A) \varphi_{\varepsilon_1 \cdots \varepsilon_N, m_1 \cdots m_N} = \sum_{\varepsilon'_k} \varphi_{\varepsilon_1 \cdots \varepsilon'_k \cdots \varepsilon_N, m_1 \cdots m_N} A_{\varepsilon'_k \varepsilon_k}$$

where $A v_\varepsilon = \sum_{\varepsilon'} v_{\varepsilon'} A_{\varepsilon' \varepsilon}$. The Yang-Baxter equation for $\tilde{R}_{j,k}(z) = (\pi_j \otimes \pi_k)\Big(\tilde{R}(z)\Big)$ then reads as follows.

$$\begin{aligned}
&\tilde{R}_{k,k+1}(z_2/z_1) \tilde{R}_{k+1,k+2}(z_3/z_1) \tilde{R}_{k,k+1}(z_3/z_2) \\
&= \tilde{R}_{k+1,k+2}(z_3/z_2) \tilde{R}_{k,k+1}(z_3/z_1) \tilde{R}_{k+1,k+2}(z_2/z_1).
\end{aligned}$$

Proposition 3 *The following relations hold:*

Commutation Relation

$$\begin{aligned}
&\varphi_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_{j+1}, z_j, \cdots, z_N) \\
&= \tilde{R}_{j,j+1}(z_{j+1}/z_j) \varphi_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_j, z_{j+1}, \cdots, z_N).
\end{aligned} \tag{2.8}$$

Fusion Relation

$$\begin{aligned}
&\varphi_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N)\Big|_{z_{j+1}=q^{-2}z_j} \\
&= (-q)^{N-j+(\varepsilon_j-1)/2} \delta_{\varepsilon_j+\varepsilon_{j+1},0} \prod_{i=1}^{j-1}(z_i - q^2 z_j) \prod_{i=j+2}^{N}(q^{-2}z_j - q^2 z_i) \\
&\quad \times \varphi_{\varepsilon_1, \cdots, \varepsilon_{j-1}, \varepsilon_{j+2}, \cdots, \varepsilon_N}(z_1, \cdots, z_{j-1}, z_{j+2}, \cdots, z_N).
\end{aligned} \tag{2.9}$$

Highest Weight Condition $\varphi_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N)|0\rangle$ *is a power series in* z_N.

The first two are direct consequences of the corresponding properties[21,12,24] of $\tilde{\Phi}^*_\varepsilon(z)$, and the last one is obvious. We remark that, in the literature, the commutation relations are written in the sense of (analytic continuation of) matrix elements. Thanks to the analyticity properties of type I vertex operators, they are valid also as formal series (see the remark at the end of A.4 in Ref.[19]). It is also possible to verify these relations directly using bosonization (see Ref.[24]).

We end this section with a lemma, to be used later.

Lemma 4 *If* $\max(m_1, \cdots, m_N) + r > 0$*, then*

$$\varphi_{\varepsilon_1\cdots\varepsilon_N, m_1\cdots m_N} \mathcal{H}_r = 0. \tag{2.10}$$

Proof. Suppose that $m_j = \max(m_1, \cdots, m_N)$. If $j = N$ then $m_N + r > 0$ and (2.10) follows. If $j < N$, then by applying (2.8) for $k = j$ we can rewrite the left hand side in the form

$$\begin{aligned} &\varphi_{\varepsilon_1\cdots\varepsilon_j\varepsilon_{j+1}\cdots\varepsilon_N, m_1\cdots m_j m_{j+1}\cdots m_N} \\ &= \sum_{\substack{\varepsilon'_j, \varepsilon'_{j+1} \\ k\geq 0}} \varphi_{\varepsilon_1\cdots\varepsilon'_j\varepsilon'_{j+1}\cdots\varepsilon_N, m_1\cdots, m_{j+1}-k, m_j+k, \cdots m_N} c^k_{\varepsilon'_j\varepsilon'_{j+1}, \varepsilon_j\varepsilon_{j+1}}. \end{aligned}$$

Since $\max(m_1, \cdots, m_{j+1} - k, m_j + k, \cdots, m_N) = m_j + k$, the statement follows by induction. $\square$

Remark. Taking $r = 0$ we find that $\varphi_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N)|0\rangle$ is actually a power series in $z_1, \cdots, z_N$.

3. Basis of level-1 modules

3.1. Completion of $\mathcal{V}_N$

In order to study the kernel of ρ_0, we deal with a generating series corresponding to (2.7), whose coefficients are certain infinite sums of $v_{\varepsilon_1, n_1} \otimes \cdots \otimes v_{\varepsilon_N, n_N}$. For the precise formulation we introduce a completion of $\mathcal{V}_N = V_{\mathrm{aff}}^{\otimes N}$.

For each $r \in \mathbb{Z}$, let

$$\mathcal{V}_N^{(r)} = \mathrm{span}_{\mathbb{C}}\{v_{\varepsilon_1, m_1} \otimes \cdots \otimes v_{\varepsilon_N, m_N} \mid m_1 + \cdots + m_N = r\}.$$

Setting

$$D(m_1, \cdots, m_N) = \max(m_1 + m_2 + \cdots + m_N, \cdots, m_{N-1} + m_N, m_N)$$

we define a decreasing filtration

$$\begin{aligned} &\mathcal{V}_N^{(r)} \supset \cdots \supset \mathcal{V}_N^{(r)}[l] \supset \mathcal{V}_N^{(r)}[l+1] \supset \cdots, \\ &\mathcal{V}_N^{(r)}[l] = \mathrm{span}_{\mathbb{C}}\{v_{\varepsilon_1, m_1} \otimes \cdots \otimes v_{\varepsilon_N, m_N} \in \mathcal{V}_N^{(r)} \mid D(m_1, \cdots, m_N) \geq l\}. \end{aligned}$$

Denote by $\widehat{\mathcal{V}}_N^{\prime(r)}$ the completion of $\mathcal{V}_N^{(r)}$ with respect to this filtration.

$$\widehat{\mathcal{V}}_N^{\prime(r)} = \varprojlim_l \mathcal{V}_N^{(r)} / \mathcal{V}_N^{(r)}[l].$$

We set $\widehat{\mathcal{V}}'_N = \oplus_{r\in\mathbb{Z}} \widehat{\mathcal{V}}_N^{\prime(r)}$, $\widehat{\mathcal{V}}' = \oplus_{N\geq 0} \widehat{\mathcal{V}}'_N$. Since $\tilde{\Phi}^*_{\varepsilon,n}\mathcal{H}_s \subset \mathcal{H}_{s+n}$ and $\mathcal{H}_s = 0$ for $s > 0$, we have $\rho\left(\mathcal{V}_N^{(r)}[l]\right)\mathcal{H}_s = 0$ if $l + s > 0$. It follows that the map ρ, and hence ρ_0, extends to the completion:

$$\widehat{\rho}' \ : \ \widehat{\mathcal{V}}' \longrightarrow \mathrm{End}_{\mathbb{C}}\mathcal{H}, \qquad \widehat{\rho}'_0 \ : \ \widehat{\mathcal{V}}' \longrightarrow \mathcal{H}.$$

3.2. Generating series $F_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N)$

Let p_j and $\eta(z)$ be as in (2.7). We define an analogous generating series whose coefficients are in $\widehat{\mathcal{V}}'_N$:

$$\begin{aligned} F_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N) &= \frac{\prod_{j=1}^N z_j^{(N-j-p_j+p_N)/2}}{\prod_{j<k}\eta(z_k/z_j)} v_{\varepsilon_1}(z_1)\otimes\cdots\otimes v_{\varepsilon_N}(z_N) \\ &= \sum_{m_1,\cdots,m_N\in\mathbb{Z}} F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N} z_1^{-m_1}\cdots z_N^{-m_N}. \end{aligned} \tag{3.1}$$

If we set

$$\begin{aligned} \mathbb{Z}_{+,N} &= \{\sum_{k=1}^{N-1} n_k(0,\cdots,\underbrace{-1,1}_{k,k+1},\cdots,0) \mid n_1,\cdots,n_{N-1}\in\mathbb{Z}_{\geq 0}\}, \\ \kappa &= (\kappa_1,\cdots,\kappa_N), \qquad \kappa_j=(N-j-p_j+p_N)/2, \end{aligned}$$

then the Laurent coefficients have the form

$$F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N} = \sum_{\substack{(n_1,\cdots,n_N)\in \\ (m_1,\cdots,m_N)+\kappa+\mathbb{Z}_{+,N}}} c_{n_1,\cdots,n_N} v_{\varepsilon_1,n_1}\otimes\cdots\otimes v_{\varepsilon_N,n_N}$$

for some $c_{n_1,\cdots,n_N}\in\mathbb{C}$. Since

$$D(m_1,\cdots,m_j-1,m_{j+1}+1,\cdots,m_N)\geq D(m_1,\cdots,m_j,m_{j+1},\cdots,m_N),$$

the sum $F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N}$ converges to an element of $\widehat{\mathcal{V}}'_N[l]$ with $l=D(m_1,\cdots,m_N)$. Conversely

$$v_{\varepsilon_1,n_1}\otimes\cdots\otimes v_{\varepsilon_N,n_N} = \sum_{\substack{(m_1,\cdots,m_N) \\ \in(n_1,\cdots,n_N)-\kappa+\mathbb{Z}_{+,N}}} \tilde{c}_{m_1,\cdots,m_N} F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N}$$

for some $\tilde{c}_{m_1,\cdots,m_N}\in\mathbb{C}$.

3.3. Second completion

By the definition we have

$$\widehat{\rho}'\left(F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N}\right) = \varphi_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N} \quad \in \mathrm{End}_{\mathbb{C}}\mathcal{H}.$$

Therefore the commutation relation (2.8) entails

Proposition 5 *The Laurent coefficients of the following belong to* $\mathrm{Ker}\,\widehat{\rho}'$.

$$\begin{aligned} &F_{\varepsilon_1,\cdots,\varepsilon_j,\varepsilon_{j+1},\cdots,\varepsilon_N}(z_1,\cdots,z_{j+1},z_j,\cdots,z_N) \\ &\quad -\tilde{R}_{j,j+1}(z_{j+1}/z_j) F_{\varepsilon_1,\cdots,\varepsilon_j,\varepsilon_{j+1},\cdots,\varepsilon_N}(z_1,\cdots,z_j,z_{j+1},\cdots,z_N). \end{aligned} \tag{3.2}$$

Let us introduce the second filtration in $\widehat{\mathcal{V}}'^{(r)}_N$.

$$\widehat{\mathcal{V}}'^{(r)}_N \supset \cdots \supset \widehat{\mathcal{V}}'^{(r)}_N[[m]] \supset \widehat{\mathcal{V}}'^{(r)}_N[[m+1]] \supset \cdots,$$
$$\widehat{\mathcal{V}}'^{(r)}_N[[m]] = \mathrm{cl.span}_{\mathbb{C}}\{F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N} \in \widehat{\mathcal{V}}'^{(r)}_N \mid \max(m_1,\cdots,m_N) \geq m\,\}$$

where cl stands for the closure in $\widehat{\mathcal{V}}'$. We denote by $\widehat{\mathcal{V}}^{(r)}_N$ the completion of $\widehat{\mathcal{V}}'^{(r)}_N$ with respect to this filtration

$$\widehat{\mathcal{V}}^{(r)}_N = \lim_{\substack{\longleftarrow \\ m}} \widehat{\mathcal{V}}'^{(r)}_N / \widehat{\mathcal{V}}'^{(r)}_N[[m]],$$

and set $\widehat{\mathcal{V}}_N = \oplus_{r\in\mathbb{Z}}\widehat{\mathcal{V}}^{(r)}_N$, $\widehat{\mathcal{V}} = \oplus_{N\geq 0}\widehat{\mathcal{V}}_N$.

In view of Lemma 4 and Proposition 5, we see that the maps $\widehat{\rho}'$ and $\widehat{\rho}'_0$ extend further to $\widehat{\mathcal{V}}$:

$$\widehat{\rho} \;:\; \widehat{\mathcal{V}} \longrightarrow \mathrm{End}_{\mathbb{C}}\mathcal{H}, \qquad \widehat{\rho}_0 \;:\; \widehat{\mathcal{V}} \longrightarrow \mathcal{H}.$$

After preparing the second completion we can state (see (2.9))

Proposition 6 *The Laurent coefficients of the following belong to* Ker $\widehat{\rho}$.

$$\begin{aligned}
& F_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N)\Big|_{z_{j+1}=q^{-2}z_j} \\
& \quad -(-q)^{N-j+(\varepsilon_j-1)/2}\delta_{\varepsilon_j+\varepsilon_{j+1},0}\prod_{i=1}^{j-1}(z_i-q^2z_j)\prod_{i=j+2}^{N}(q^{-2}z_j-q^2z_i) \\
& \qquad \times F_{\varepsilon_1,\cdots,\varepsilon_{j-1},\varepsilon_{j+2},\cdots,\varepsilon_N}(z_1,\cdots,z_{j-1},z_{j+2},\cdots,z_N).
\end{aligned} \tag{3.3}$$

3.4. Kernel of $\widehat{\rho}_0$

From the remark at the end of 2.4 we see that Ker $\widehat{\rho}_0$ contains

$$F_{\varepsilon_1\cdots\varepsilon_N,m_1\cdots m_N} \quad \text{with } m_j > 0 \text{ for some } j. \tag{3.4}$$

The Laurent coefficients of (3.2) and (3.3) reduce to finite sums modulo Ker $\widehat{\rho}_0$, and therefore Ker ρ_0 contains (3.2) and (3.3) in reduced form. Thus, it follows that

$$\widehat{\mathcal{V}}/\mathrm{Ker}\,\widehat{\rho}_0 \simeq \widehat{\mathcal{V}}'/\mathrm{Ker}\,\widehat{\rho}'_0 \simeq \mathcal{V}/\mathrm{Ker}\,\rho_0 \simeq \mathcal{H}.$$

Let $\mathcal{N} \subset \widehat{\mathcal{V}}$ be (the closure of) the span of elements (3.2), (3.3) and (3.4).

Proposition 7

$$\widehat{\mathcal{V}}/\mathcal{N} \simeq \mathcal{H}.$$

This can be verified as follows. As we will show in Proposition 8, the general expression $v_{\varepsilon_1,n_1}\otimes\cdots\otimes v_{\varepsilon_N,n_N}$ can be reduced modulo $\mathcal{N}$ to a certain normal ordered form. We then count the number of such normal ordered expressions to find that the character of $\widehat{\mathcal{V}}/\mathcal{N}$ is dominated by that of $\mathcal{H}$. Since $\widehat{\rho}_0 : \widehat{\mathcal{V}} \to \mathcal{H}$ is surjective, this proves the proposition. In fact this working has already been discussed in the paper[19].

(Note that the properties of the elliptic algebra assumed in Ref.[19] are all valid for quantum affine algebras.) Hence it suffices to show that the 'normal ordering rules' of Ref.[19] are consequences of (3.2) and (3.3).

Define

$$\overline{F}_{\varepsilon_1,\cdots,\varepsilon_N}(\zeta_1,\cdots,\zeta_N)=\frac{\prod_{j=1}^N \zeta_j^{(1+\varepsilon_j)/2}}{\prod_{j=1}^N z_j^{N-j}\prod_{j<k}(1-q^2z_k/z_j)}F_{-\varepsilon_1\cdots-\varepsilon_N}(z_1,\cdots,z_N) \tag{3.5}$$

where $z_j=\zeta_j^2$. Note that the coefficients of (3.5) are well–defined in $\widehat{\mathcal{V}}$.

Proposition 8 *The following belong to* $\mathrm{Ker}\,\widehat{\rho}$.

$$\begin{aligned}&\overline{F}_{\varepsilon_1\cdots\underbrace{\varepsilon\varepsilon}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_{k+1},\zeta_k,\cdots,\zeta_N)\\ &\qquad -\overline{F}_{\varepsilon_1\cdots\underbrace{\varepsilon\varepsilon}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&(\zeta_{k+1}+q\zeta_k)\Big(\overline{F}_{\varepsilon_1\cdots\underbrace{+-}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_{k+1},\zeta_k,\cdots,\zeta_N)\\ &\qquad +\overline{F}_{\varepsilon_1\cdots\underbrace{-+}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_{k+1},\zeta_k,\cdots,\zeta_N)\Big)\\ &-(\zeta_k+q\zeta_{k+1})\Big(\overline{F}_{\varepsilon_1\cdots\underbrace{+-}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N)\\ &\qquad +\overline{F}_{\varepsilon_1\cdots\underbrace{-+}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N)\Big).\end{aligned} \tag{3.7}$$

These relations are precisely the normal ordering rules in Ref.[19].

Proof. Let us write $A\sim B$ to mean that $A-B\in\mathrm{Ker}\,\widehat{\rho}$. Then (3.6) is shown as follows.

$$\begin{aligned}&\overline{F}_{\varepsilon_1\cdots\underbrace{\varepsilon\varepsilon}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_{k+1},\zeta_k,\cdots,\zeta_N)\\ &=\frac{\prod_{j=1}^N \zeta_j^{(1+\varepsilon_j)/2}}{\prod_{j=1}^N z_j^{N-j}\prod_{j<k}(1-q^2z_k/z_j)}\frac{z_k(1-q^2z_{k+1}/z_k)}{z_{k+1}(1-q^2z_k/z_{k+1})}\\ &\qquad\times F_{-\varepsilon_1\cdots\underbrace{-\varepsilon,-\varepsilon}_{k,k+1}\cdots-\varepsilon_N}(z_1,\cdots,z_{k+1},z_k,\cdots,z_N)\\ &\sim\frac{\prod_{j=1}^N \zeta_j^{(1+\varepsilon_j)/2}}{\prod_{j=1}^N z_j^{N-j}\prod_{j<k}(1-q^2z_k/z_j)}\\ &\qquad\times F_{-\varepsilon_1\cdots\underbrace{-\varepsilon,-\varepsilon}_{k,k+1}\cdots-\varepsilon_N}(z_1,\cdots,z_k,z_{k+1},\cdots,z_N)\\ &=\overline{F}_{\varepsilon_1\cdots\underbrace{\varepsilon\varepsilon}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\overline{F}_{\varepsilon_1\cdots\underbrace{\pm\mp}_{k,k+1}\cdots\varepsilon_N}(\zeta_1,\cdots,\zeta_{k+1},\zeta_k,\cdots,\zeta_N)\\
&\sim\frac{(1-q^2)\zeta_k/\zeta_{k+1}}{1-q^2z_k/z_{k+1}}\overline{F}_{\varepsilon_1\cdots\underbrace{\mp\pm}_{k,k+1}\cdots\varepsilon_N}(\zeta_k,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N)\\
&+\frac{q(1-z_k/z_{k+1})}{1-q^2z_k/z_{k+1}}\overline{F}_{\varepsilon_1\cdots\underbrace{\pm\mp}_{k,k+1}\cdots\varepsilon_N}(\zeta_k,\cdots,\zeta_k,\zeta_{k+1},\cdots,\zeta_N).
\end{aligned}$$

The statement (3.7) follows from this. □

3.5. Hecke algebra

In section 4 we will define a 'new' level-0 action of U on the level-1 module $\mathcal{H}$. For this purpose we need to rewrite the commutation relation (3.2) in the language of Hecke algebras.

Define $S\in\mathrm{End}_{\mathbb{C}}V\otimes V$ by

$$\begin{aligned}
Sv_\varepsilon\otimes v_\varepsilon&=-q^{-1}v_\varepsilon\otimes v_\varepsilon,\\
Sv_+\otimes v_-&=(q-q^{-1})v_+\otimes v_--v_-\otimes v_+,\\
Sv_-\otimes v_+&=-v_+\otimes v_-.
\end{aligned}$$

The operators $S_{j,j+1}\in\mathrm{End}_{\mathbb{C}}V^{\otimes N}$ $(j=1,\cdots,N-1)$ satisfy the Hecke algebra relation

$$S_{j,j+1}-S_{j,j+1}^{-1}=q-q^{-1}, \tag{3.8}$$

$$S_{j,j+1}S_{k,k+1}=S_{k,k+1}S_{j,j+1}\qquad(|j-k|>1), \tag{3.9}$$

$$S_{j,j+1}S_{j+1,j+2}S_{j,j+1}=S_{j+1,j+2}S_{j,j+1}S_{j+1,j+2}. \tag{3.10}$$

The R-matrix $\tilde{R}(z)$ can be written as

$$\tilde{R}(z)=\frac{Sz-S^{-1}}{qz-q^{-1}}. \tag{3.11}$$

Substituting (3.11) in (3.2) we obtain formally the relations of the form

$$(S_{j,j+1}-G_{j,j+1})F_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N)\sim 0, \tag{3.12}$$

where

$$G_{j,k}^{\pm1}=\frac{q^{-1}z_j-qz_k}{z_j-z_k}(K_{j,k}-1)+q^{\pm1} \tag{3.13}$$

and $K_{j,k}$ signifies the exchange of variables z_j and z_k. The $G_{j,j+1}$ $(j=1,\cdots,N-1)$ satisfy the same relations (3.8)–(3.10) as do $S_{j,j+1}$.

The operators $G_{j,k}$ and $K_{j,k}$ are acting on formal Laurent series $f(z_1,\cdots,z_N)$ in $z_1,\cdots,z_N$, and preserve $\mathbb{C}[z_1,z_1^{-1},\cdots,z_N,z_N^{-1}]$. However, even if the coefficients of $f(z_1,\cdots,z_N)$ all belong to $\widehat{\mathcal{V}}$, those of $G_{j,k}f(z_1,\cdots,z_N)$ no longer do so in general. In order to make sense of (3.12) we need some consideration given below.

3.6. Local series

Consider a series in $(z_1,\cdots,z_N)$ with coefficients in $\widehat{\mathcal{V}}_N$,

$$f(z_1,\cdots,z_N)=\sum_{m_1,\cdots,m_N\in\mathbb{Z}} f_{m_1,\cdots,m_N} z_1^{-m_1}\cdots z_N^{-m_N}, \tag{3.14}$$

where $f_{m_1,\cdots,m_N}\in\widehat{\mathcal{V}}_N$. We call $f(z_1,\cdots,z_N)$ *local* and homogeneous if there exists an $m\in\mathbb{Z}$ such that

$$f_{m_1,\cdots,m_N}\in\widehat{\mathcal{V}}_N^{(m_1+\cdots m_N)}[[m+\max(m_1,\cdots,m_N)]], \tag{3.15}$$

for all $m_1,\cdots,m_N$. In general, we call $f(z_1,\cdots,z_N)$ local if it can be written as a finite sum of local homogeneous series. The generating series $F_{\varepsilon_1,\cdots,\varepsilon_N}(z_1,\cdots,z_N)$ (3.1) is local in this sense.

An important property of local series is the following:

Lemma 9 *Consider an arbitrary Laurent series homogeneous in $z_1,\cdots,z_N$:*

$$c(z_1,\cdots,z_N)=\sum_{\substack{k_1,\cdots,k_N\in\mathbb{Z}\\ k_1+\cdots+k_N=l}} c_{k_1,\cdots,k_N} z_1^{k_1}\cdots z_N^{k_N}.$$

If $f(z_1,\cdots,z_N)$ is local, then the product $c(z_1,\cdots,z_N)f(z_1,\cdots,z_N)$ is well-defined in $\widehat{\mathcal{V}}$.

Proof. Expand the product

$$\begin{aligned}&c(z_1,\cdots,z_N)f(z_1,\cdots,z_N)\\ &=\sum_{m_1,\cdots,m_N}\left(\sum_{\substack{k_1,\cdots,k_N\\ k_1+\cdots+k_N=l}} c_{k_1,\cdots,k_N} f_{m_1+k_1\cdots m_N+k_N}\right) z_1^{-m_1}\cdots z_N^{-m_N}.\end{aligned}$$

We must show that each coefficient

$$f'_{m_1\cdots m_N}=\sum_{\substack{k_1,\cdots,k_N\\ k_1+\cdots+k_N=l}} c_{k_1,\cdots,k_N} f_{m_1+k_1\cdots m_N+k_N}$$

is convergent in $\widehat{\mathcal{V}}_N$. This holds since there are only finitely many k_j's such that $\max(m_1+k_1,\cdots,m_N+k_N)$ is bounded from above and $k_1+\cdots+k_N=l$. □

In particular, we can multiply any homogeneous element in the ring of Laurent series that are convergent on the unit circles $|z_1|=\cdots=|z_N|=1$. We note however that the product $c(z_1,\cdots,z_N)f(z_1,\cdots,z_N)$ is not necessarily local.

Lemma 10 *If $f(z_1, \cdots, z_N)$ is local, then $G_{j,k} f(z_1, \cdots, z_N)$ is well-defined and local.*

Proof. Applying $G_{j,k}$ to a monomial we obtain an expression

$$G_{j,k} z_j^{-m_j} z_k^{-m_k} = \sum c_{n_j n_k}^{m_j m_k} z_j^{-n_j} z_k^{-n_k}$$

where the sum is taken over n_j, n_k such that $n_j + n_k = m_j + m_k$ and $\max(n_j, n_k) \leq \max(m_j, m_k)$. Therefore if (3.14) is local and homogeneous, then the coefficients of $G_{j,k} f(z_1, \cdots, z_N)$ have the form

$$f'_{n_1 \cdots n_N} = \sum_{m_j, m_k} c_{n_j n_k}^{m_j m_k} f_{n_1, \cdots, m_j, \cdots, m_k, \cdots, n_N}$$

where $m_j + m_k = n_j + n_k$ and $\max(m_j, m_k) \geq \max(n_j, n_k)$. The lemma follows from this observation. $\square$

3.7. The '$S = G$' relations

Let us return to the relations (3.12).

Proposition 11 *The relations (3.2) are equivalent to*

$$(S_{j,j+1} - G_{j,j+1}) F_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N) \sim 0. \tag{3.16}$$

Proof. Write (3.2) as

$$\begin{aligned} &F_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_{j+1}, z_j, \cdots, z_N) \\ &\sim \tilde{R}_{j,j+1}(z_{j+1}/z_j) F_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_j, z_{j+1}, \cdots, z_N). \end{aligned} \tag{3.17}$$

Multiplying $q z_{j+1} - q^{-1} z_j$ we obtain

$$\begin{aligned} &(q z_{j+1} - q^{-1} z_j) F_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_{j+1}, z_j, \cdots, z_N) \\ &\sim (S_{j,j+1} z_{j+1} - S_{j,j+1}^{-1} z_j) F_{\varepsilon_1, \cdots, \varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_N}(z_1, \cdots, z_j, z_{j+1}, \cdots, z_N). \end{aligned} \tag{3.18}$$

Conversely, by multiplying the Laurent series $1/(q z_{j+1} - q^{-1} z_j)$, which is convergent for $|z_j| = |z_{j+1}| = 1$, we get (3.17) from (3.18). Thus (3.17) and (3.18) are equivalent.

The relation (3.18) is equivalent to

$$(z_{j+1} - z_j)\Big((S_{j,j+1} - G_{j,j+1}) F_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N)\Big) \sim 0. \tag{3.19}$$

Set

$$(S_{j,j+1} - G_{j,j+1}) F_{\varepsilon_1, \cdots, \varepsilon_N}(z_1, \cdots, z_N) = \sum_{m_1, \cdots, m_N \in \mathbb{Z}} h_{m_1, \cdots, m_N} z_1^{-m_1} \cdots z_N^{-m_N}. \tag{3.20}$$

Then (3.20) is local, and we have

$$h_{m_1, \cdots, m_j, m_{j+1}, \cdots, m_N} \sim h_{m_1, \cdots, m_j - 1, m_{j+1} + 1, \cdots, m_N}.$$

Using this repeatedly, we find $h_{m_1,\cdots,m_N} \sim 0$. $\square$

4. New action

4.1. Affine Hecke algebras

The action of $U_q^{\rm op}(\mathfrak{sl}_2)$ on each $\mathcal{V}_N = V_{\rm aff}^{\otimes N}$ extends naturally to the completion $\hat{\mathcal{V}}$. The subspace $\mathcal{N} \subset \hat{\mathcal{V}}$, given as the span of (3.16), (3.3) and (3.4), is invariant with respect to this action. This can be seen explicitly by noting that $[S, \Delta^{\rm op}(x)] = 0$ $(\forall x \in U_q(\mathfrak{sl}_2))$ and that the vector

$$v_+ \otimes v_- - q^{-1} v_- \otimes v_+ \tag{4.1}$$

belongs to the trivial representation of $U_q(\mathfrak{sl}_2)$. Hence $\mathcal{H} \simeq \widehat{\mathcal{V}}/\mathcal{N}$ is an $U_q(\mathfrak{sl}_2)$-module. We will extend this to the level-0 action of the quantum affine algebra U.

Recall that, given complex numbers $a_j \neq 0$ $(j = 1, \cdots, N)$, one can extend the $U_q(\mathfrak{sl}_2)$-module $\mathcal{V}_N$ to the evaluation module over U (denoted by $\pi_{a_1,\cdots,a_N}$)

$$\pi_{a_1,\cdots,a_N}(e_0) = \sum_{j=1}^{N} a_j \pi_j(f_1) \pi_{j+1}(t_1^{-1}) \cdots \pi_N(t_1^{-1}), \tag{4.2}$$

$$\pi_{a_1,\cdots,a_N}(f_0) = \sum_{j=1}^{N} a_j^{-1} \pi_1(t_1) \cdots \pi_{j-1}(t_1) \pi_j(e_1), \tag{4.3}$$

$$\pi_{a_1,\cdots,a_N}(t_0) = \pi_1(t_1^{-1}) \cdots \pi_N(t_1^{-1}). \tag{4.4}$$

However this action does not descend to $\mathcal{H}$ because it violates the condition $e_0\mathcal{N} \subset \mathcal{N}$ and $f_0\mathcal{N} \subset \mathcal{N}$. The remedy is to replace the numbers a_j by suitable commuting operators, to be given below.

Following Ref.[3] let us introduce the operators

$$\begin{aligned} Y_j &= (G_{j,j+1}^{-1} K_{j,j+1}) \cdots (G_{j,N}^{-1} K_{j,N}) p^{\vartheta_j} (K_{1,j} G_{1,j}) \cdots (G_{j-1,j} K_{j-1,j}) \\ &= G_{j,j+1}^{-1} \cdots G_{N-1,N}^{-1} Z G_{1,2} \cdots G_{j-1,j}, \end{aligned} \tag{4.5}$$

where $Z = K_{1,2} K_{1,3} \cdots K_{1,N} p^{\vartheta_1}$ and p^{ϑ_j} denotes the scale operator

$$p^{\vartheta_j} f(z_1, \cdots, z_N) = f(z_1, \cdots, p z_j, \cdots, z_N).$$

At this stage the parameter p is arbitrary, but we will make a specific choice of it later on.

The operators $G_{j,j+1}$ $(j = 1, \cdots, N-1)$ and Y_j $(j = 1, \cdots, N)$ are known to satisfy the relations for the affine Hecke algebra $\hat{H}_N$. Namely we have, in addition to (3.8)–(3.10),

$$\begin{aligned} &Y_j Y_k = Y_k Y_j, \\ &G_{j,j+1} Y_j G_{j,j+1} = Y_{j+1}, \\ &[G_{j,j+1}, Y_k] = 0, \qquad (j, j+1 \neq k). \end{aligned}$$

In what follows, for an operator $X \in \mathrm{End}_{\mathbb{C}}\mathbb{C}[z_1, z_1^{-1}, \cdots, z_N, z_N^{-1}]$ we define $\hat{X} \in \mathrm{End}_{\mathbb{C}}\hat{\mathcal{V}}_N$ by setting

$$\sum \left(\hat{X} F_{\varepsilon_1\cdots\varepsilon_N; m_1,\cdots,m_N}\right) z_1^{-m_1} \cdots z_N^{-m_N} = X F_{\varepsilon_1\cdots\varepsilon_N}(z_1, \cdots, z_N).$$

Notice that $\widehat{XY} = \hat{Y}\hat{X}$.

We now define $\pi^{(N)}(e_0)$, $\pi^{(N)}(f_0)$ by the right hand side of (4.2) and (4.3) respectively, wherein we set

$$a_j = q^{N-1}\hat{Y}_j^{-1}. \tag{4.6}$$

The following says that they leave the subspace $\mathcal{N}$ invariant.

Proposition 12 *Let $p = q^4$. For $x = e_0$ or f_0, we have*

$$\pi^{(N)}(x)\Big(S_{j,j+1} - G_{j,j+1}\Big)F_{\varepsilon_1\cdots\varepsilon_N}(z_1, \cdots, z_N) \sim 0, \tag{4.7}$$

$$\begin{aligned}
&\Big(\pi^{(N)}(x)F_{\varepsilon_1\cdots\varepsilon_N}(z_1, \cdots, z_j, z_{j+1}, \cdots, z_N)\Big)\Big|_{z_{j+1}=q^{-2}z_j} \\
&\qquad \sim (-q)^{N-j+(\varepsilon_j-1)/2}\delta_{\varepsilon_j+\varepsilon_{j+1},0}\prod_{i=1}^{j-1}(z_i - q^2 z_j)\prod_{i=j+2}^{N}(q^{-2}z_j - q^2 z_i) \\
&\times\pi^{(N-2)}(x)F_{\varepsilon_1\cdots\varepsilon_N}(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_N),
\end{aligned} \tag{4.8}$$

$\pi^{(N)}(x)F_{\varepsilon_1\cdots\varepsilon_N,m_1,\cdots,m_N}$ belongs to the kernel of $\widehat{\rho}_0$

if $m_j > 0$ for some j. (4.9)

This proposition gives us

Theorem 13 *The formulas (4.2)–(4.4) with the choice (4.6), (4.5) and $p = q^4$ defines a level-0 action of U on $\mathcal{H}$.*

The rest of the text is devoted to the proof of Proposition 12. The statement (4.9) is valid because the operator Y_j keeps the spce $\mathbb{C}[z_1^{-1}, \cdots, z_N^{-1}]$ invariant and preserves the degree.

4.2. Proof of (4.7)

For this the parameter p can be arbitrary. The way of verification is the same as in Ref.[10]. Consider the case $x = e_0$ and set

$$\begin{aligned}
e_0^{(k)} &= q^{N-1}\hat{Y}_k^{-1}f^{(k)}, \\
f^{(k)} &= \pi_k(f_1)\pi_{k+1}(t_1^{-1})\cdots\pi_N(t_1^{-1}).
\end{aligned}$$

Clearly $e_0^{(k)}$ commutes with $G_{j,j+1}$ and $S_{j,j+1}$ if $k \neq j, j+1$. Using

$$S(f_1 \otimes t_1^{-1}) = (1 \otimes f_1)S, \tag{4.10}$$

and $[\hat{Y}_k^{-1}, S_{j,j+1}] = 0$, we have

$$\begin{aligned}
e_0^{(j)}S_{j,j+1} &= e_0^{(j)}(S_{j,j+1}^{-1} + q - q^{-1}) \\
&= S_{j,j+1}^{-1}q^{N-1}\hat{Y}_j^{-1}f^{(j+1)} + (q - q^{-1})e_0^{(j)}, \\
e_0^{(j+1)}S_{j,j+1} &= S_{j,j+1}q^{N-1}\hat{Y}_{j+1}^{-1}f^{(j)}
\end{aligned}$$

On the other hand, from $\hat{G}_{j,j+1}\hat{Y}_j\hat{G}_{j,j+1}=\hat{Y}_{j+1}$ and $[\hat{G}_{j,j+1},f^{(k)}]=0$, we find

$$\begin{aligned} e_0^{(j)}\hat{G}_{j,j+1} &= e_0^{(j)}(\hat{G}_{j,j+1}^{-1}+q-q^{-1}) \\ &= \hat{G}_{j,j+1}q^{N-1}\hat{Y}_{j+1}^{-1}f^{(j)}+(q-q^{-1})e_0^{(j)}, \\ e_0^{(j+1)}\hat{G}_{j,j+1} &= \hat{G}_{j,j+1}^{-1}q^{N-1}\hat{Y}_j^{-1}f^{(j+1)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & (e_0^{(j)}+e_0^{(j+1)})\Big(S_{j,j+1}-\hat{G}_{j,j+1}\Big) \\ = & \Big(S_{j,j+1}^{-1}-\hat{G}_{j,j+1}^{-1}\Big)q^{N-1}\hat{Y}_j^{-1}f^{(j+1)} \\ & \qquad +\Big(S_{j,j+1}-\hat{G}_{j,j+1}\Big)q^{N-1}\hat{Y}_{j+1}^{-1}f^{(j)} \\ = & (S_{j,j+1}-\hat{G}_{j,j+1})q^{N-1}(\hat{Y}_j^{-1}f^{(j+1)}+\hat{Y}_{j+1}^{-1}f^{(j)}). \end{aligned}$$

Hence

$$\begin{aligned} & \pi^{(N)}(e_0)(S_{j,j+1}-\hat{G}_{j,j+1})F_{\varepsilon_1\cdots\varepsilon_N}(z_1,\cdots,z_N)= \\ & \quad (S_{j,j+1}-\hat{G}_{j,j+1})(\sum_{k(\neq j,j+1)}e_0^{(k)}+\hat{Y}_j^{-1}f^{(j+1)}+\hat{Y}_{j+1}^{-1}f^{(j)}) \\ & \qquad\qquad \times F_{\varepsilon_1\cdots\varepsilon_N}(z_1,\cdots,z_N) \quad \sim \quad 0, \end{aligned}$$

which completes the proof for e_0.

The case $x=f_0$ can be shown similarly by using

$$S(t_1\otimes e_1)=(e_1\otimes 1)S.$$

4.3. Proof of (4.8)

The verification of (4.8) is technically more complicated. For this it is necessary to choose $p=q^4$. In view of (3.2) it suffices to consider the case $j=N-1$. The case $x=f_0$ being similar, we concentrate on the case $x=e_0$.

For convenience we use the decomposition of the $U_q^{\mathrm{op}}(\mathfrak{sl}_2)$-module $V\otimes V=V^{(3)}\oplus V^{(1)}$, the superscripts indicating the dimensions. For $s=1$ or 3 let $v^{(s)}=\sum_{\varepsilon,\varepsilon'}v_\varepsilon\otimes v_{\varepsilon'}c^{(s)}_{\varepsilon\varepsilon'}$ be a vector in $V^{(s)}$. We have

$$Sv^{(3)}=-q^{-1}v^{(3)}, \qquad Sv^{(1)}=qv^{(1)}. \tag{4.11}$$

Writing $\sum_{\varepsilon,\varepsilon'}F_{\varepsilon_1\cdots\underbrace{\varepsilon\varepsilon'}_{j,j+1}\cdots\varepsilon_N}(z_1,\cdots,z_N)c^{(s)}_{\varepsilon\varepsilon'}$ as $F_{\varepsilon_1\cdots c^{(s)}\cdots\varepsilon_N}(z_1,\cdots,z_N)$, we shall verify (4.8) separately for $s=1,3$.

The case $s=3$: From (4.5) and (4.10), we have

$$\begin{aligned} & \rho^{(N)}(e_0)F_{\varepsilon_1\cdots\varepsilon_N}(z_1,\cdots,z_N) \\ & =\sum_{j=1}^N f^{(N)}S_{N-1,N}\cdots S_{j,j+1}G_{j-1,j}^{-1}\cdots G_{1,2}^{-1}F_{\varepsilon_1\cdots\varepsilon_N}(z_2,\cdots,z_N,p^{-1}z_1). \end{aligned}$$

From (3.3) we have $F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(3)}}(z_1,\cdots,z_N)\Big|_{z_N=q^{-2}z_{N-1}}\sim 0$. Therefore, we are to show

$$\sum_{j=1}^{N} f^{(N)}S_{N-1,N}\cdots S_{j,j+1}G^{-1}_{j-1,j}\cdots G^{-1}_{1,2} \\ \times F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(3)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big|_{z_N=q^{-2}z_{N-1}}\sim 0.$$

Step 1. The sum of the terms for $j=N-1$ and $j=N$ reads

$$f^{(N)}(S_{N-1,N}+G^{-1}_{N-1,N})G^{-1}_{N-2,N-1}\cdots G^{-1}_{1,2}\times \\ \times F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(3)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big|_{z_N=q^{-2}z_{N-1}}. \tag{4.12}$$

Because of (4.11) and $G^{-1}_{N-1,N}\Big|_{z_N=q^{-2}z_{N-1}}=q^{-1}$, the sum (4.12) vanishes.

Step 2. Note that $S_{2,3}S_{1,2}(V\otimes V^{(3)})\subset V^{(3)}\otimes V$. Therefore, for $j\le N-2$ the term

$$f^{(N)}S_{N-1,N}\cdots S_{j,j+1}G^{-1}_{j-1,j}\cdots G^{-1}_{1,2}F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(3)}}(z_2,\cdots,z_N,p^{-1}z_1) \tag{4.13}$$

is a linear combination of

$$G^{-1}_{j-1,j}\cdots G^{-1}_{1,2}F_{\varepsilon'_1\cdots\varepsilon'_{N-3}c'^{(3)}\varepsilon'_N}(z_2,\cdots,z_N,p^{-1}z_1)$$

where $\sum_{\varepsilon,\varepsilon'}v_\varepsilon\otimes v_{\varepsilon'}c'^{(3)}_{\varepsilon\varepsilon'}\in V^{(3)}$. Therefore, at $z_N=q^{-2}z_{N-1}$ the term (4.13) vanishes.

The case $s=1$: Next we consider the case of $F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}$, where $v^{(1)}=\sum_{\varepsilon,\varepsilon'}v_\varepsilon\otimes v_{\varepsilon'}c^{(1)}_{\varepsilon\varepsilon'}$ is given by (4.1). Below, we give a complete proof for $N\le 6$. We have verified the general case by a similar method.

We set

$$B_{j,k}=\frac{q^{-1}z_j-qz_k}{z_j-z_k},\quad C_{j,k}=\frac{(q-q^{-1})z_j}{z_j-z_k},\quad \overline{C}_{j,k}=\frac{(q-q^{-1})z_k}{z_j-z_k}, \tag{4.14}$$

so that $G_{j,k}=B_{j,k}K_{j,k}+C_{j,k}$ and $G^{-1}_{j,k}=B_{j,k}K_{j,k}+\overline{C}_{j,k}$.

Lemma 14 $S_{2,3}S_{1,2}v_\varepsilon\otimes v^{(1)}=q^{-1}v^{(1)}\otimes v_\varepsilon$.

Lemma 15

$$F_{\varepsilon_1\cdots\underbrace{c^{(1)}}_{j,j+1}\cdots\varepsilon_N}(z_1,\cdots,z_j,z_{j+1},\cdots,z_N)\Big|_{z_{j+1}=q^{-2}z_j} \\ =(-q)^{N-j}(1+q^{-2})F_{\varepsilon_1\cdots,\varepsilon_{j-1},\varepsilon_{j+2},\cdots\varepsilon_N}(z_1,\cdots,z_{j-1},z_{j+2},\cdots,z_N)\times \\ \times\prod_{k=1}^{j-1}(z_k-q^2z_j)\prod_{k=j+2}^{N}(q^{-2}z_j-q^2z_k).$$

We are to show

$$
\begin{aligned}
&\Big(q^2 f^{(N)} \sum_{j=1}^{N} S_{N-1,N}\cdots S_{j,j+1} G^{-1}_{j-1,j}\cdots G^{-1}_{1,2} \\
&\times F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big)\Big|_{z_N=q^{-2}z_{N-1}} \\
&\sim -(q+q^{-1}) f^{(N-2)} \sum_{j=1}^{N-2} S_{N-3,N-2}\cdots S_{j,j+1} G^{-1}_{j-1,j}\cdots G^{-1}_{1,2} \\
&\times \left(F_{\varepsilon_1\cdots\varepsilon_{N-2}}(z_2,\cdots,z_{N-2},p^{-1}z_1)\prod_{k=1}^{N-2}(z_k-q^2 z_{N-1})\right).
\end{aligned} \tag{4.15}
$$

Step 1. By using Lemma 14 and Lemma 15, the relation (4.15) reduces to

$$
\begin{aligned}
&(1-q^2) z_{N-1} \sum_{j=1}^{N-2} f^{(N-2)} S_{N-3,N-2}\cdots S_{j,j+1} G^{-1}_{j-1,j}\cdots G^{-1}_{1,2}\times \\
&\times\Big\{F_{\varepsilon_1\cdots\varepsilon_{N-2}}(z_2,\cdots,z_{N-2},p^{-1}z_1)\prod_{k=1}^{N-2}(z_k-q^2 z_{N-1})\Big\} \\
&+q^2\Big\{f^{(N)} G^{-1}_{N-2,N-1}\cdots G^{-1}_{1,2}\times \\
&F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big\}\Big|_{z_N=q^{-2}z_{N-1}} \sim 0.
\end{aligned} \tag{4.16}
$$

If $N=2$, this is $0\sim 0$. If $N=3$, the left hand side is

$$
(1-q^2) z_2 f^{(1)} F_{\varepsilon_1}(p^{-1}z_1) + q^2\Big\{f^{(3)} G^{-1}_{1,2} F_{\varepsilon_1 c^{(1)}}(z_2,z_3,p^{-1}z_1)\Big\}\Big|_{z_2=q^2 z_3}. \tag{4.17}
$$

We have (see (4.14))

$$
\begin{aligned}
&f^{(3)} G^{-1}_{1,2} F_{\varepsilon_1 c^{(1)}}(z_2,z_3,p^{-1}z_1) \\
&= -q^{-1}(B_{1,2}K_{1,2}+\overline{C}_{1,2}) F_{\varepsilon_1--}(z_2,z_3,p^{-1}z_1).
\end{aligned}
$$

We have $B_{1,2}K_{1,2}F_{\varepsilon_1--}(z_2,z_3,p^{-1}z_1) = B_{1,2}F_{\varepsilon_1--}(z_1,z_3,p^{-1}z_2)$. If $z_3=q^{-2}z_2$, then we have $p^{-1}z_2=q^{-2}z_3$. (Here we used $p=q^4$.) Therefore, we have

$$
B_{1,2}K_{1,2}F_{\varepsilon_1--}(z_2,z_3,p^{-1}z_1)\Big|_{z_3=q^{-2}z_2} \sim 0.
$$

Noting that $F_{\varepsilon_1--}(z_2,z_3,p^{-1}z_1)\Big|_{z_3=q^{-2}z_2} = f^{(1)}F_{\varepsilon_1}(p^{-1}z_1)(z_2-z_1)$ and using (4.14), we obtain that (4.17) belongs to $\mathcal{N}$.

Lemma 16 $\overline{C}_{2,3}G^{-1}_{1,2} = (G^{-1}_{1,2}+C_{2,3})\overline{C}_{1,3}$.

Lemma 17

$$
\begin{aligned}
&B_{j,j+1}F_{\varepsilon_1\cdots\varepsilon_N}(z_1,\cdots,z_{j+1},z_j,\cdots,z_N) \\
&\sim (S_{j,j+1}-C_{j,j+1})F_{\varepsilon_1\cdots\varepsilon_N}(z_1,\cdots,z_j,z_{j+1},\cdots,z_N).
\end{aligned}
$$

Remark. In applying Lemma 17 more than once, we must be careful about ordering the factors. For example, for $N=5$ we have

$$\begin{aligned}&B_{3,4}B_{2,4}F_{\varepsilon_1\varepsilon_2\varepsilon_3 c^{(1)}}(z_4,z_2,z_3,z_5,p^{-1}z_1)\\&=(S_{2,3}-C_{3,4})(S_{1,2}-C_{2,4})F_{\varepsilon_1\varepsilon_2\varepsilon_3 c^{(1)}}(z_2,z_3,z_4,z_5,p^{-1}z_1).\end{aligned}$$

Step 2. By using Lemma 16 and Lemma 17, we have

$$\begin{aligned}&G^{-1}_{N-2,N-1}\cdots G^{-1}_{12}F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big|_{z_N=q^{-2}z_{N-1}}\\&=\Big\{(G^{-1}_{N-3,N-2}+C_{N-2,N-1})(G^{-1}_{N-4,N-3}+C_{N-3,N-1})\cdots(G^{-1}_{12}+C_{2,N-1})\\&+(G^{-1}_{N-4,N-3}+C_{N-3,N-1})\cdots(G^{-1}_{12}+C_{2,N-1})(S_{N-3,N-2}-C_{N-2,N-1})\\&+(G^{-1}_{N-5,N-4}+C_{N-4,N-1})\cdots(S_{N-3,N-2}-C_{N-2,N-1})(S_{N-4,N-3}-C_{N-3,N-1})\\&+\cdots\\&+(S_{N-3,N-2}-C_{N-2,N-1})(S_{N-4,N-3}-C_{N-3,N-1})\cdots(S_{12}-C_{2,N-1})\Big\}\\&\quad\times\overline{C}_{1,N-1}F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big|_{z_N=q^{-2}z_{N-1}}\\&+B_{N-2,N-1}\cdots B_{2,N-1}B_{1,N-1}F_{\varepsilon_1,\cdots,\varepsilon_{N-2},c^{(1)}}(z_1,\cdots,z_{N-2},z_N,p^{-1}z_{N-1})\Big|_{z_N=q^{-2}z_{N-1}}.\end{aligned}\tag{4.18}$$

Note that the last term vanishes when $f^{(N)}$ is applied:

$$f^{(N)}F_{\varepsilon_1,\cdots,\varepsilon_{N-2},c^{(1)}}(z_1,\cdots,z_{N-2},z_N,p^{-1}z_{N-1})\Big|_{z_N=q^{-2}z_{N-1}}\sim 0.$$

For $N=5$, the derivation of (4.18) goes as follows. By using Lemma 16 we have the equality

$$\begin{aligned}&B_{N-2,N-1}\cdots B_{2,N-1}B_{1,N-1}G^{-1}_{3,4}G^{-1}_{2,3}G^{-1}_{1,2}\\&=\overline{C}_{3,4}G^{-1}_{2,3}G^{-1}_{1,2}+B_{3,4}G^{-1}_{2,4}G^{-1}_{1,2}K_{3,4}\\&=(G^{-1}_{2,3}+C_{3,4})\overline{C}_{2,4}G^{-1}_{1,2}+B_{3,4}\overline{C}_{2,4}G^{-1}_{1,2}K_{3,4}+B_{3,4}B_{2,4}G^{-1}_{1,4}K_{2,4}K_{3,4}\\&=(G^{-1}_{2,3}+C_{3,4})(G^{-1}_{1,2}+C_{2,4})\overline{C}_{1,4}+B_{3,4}(G^{-1}_{1,2}+C_{2,4})\overline{C}_{1,4}K_{3,4}\\&+B_{3,4}B_{2,4}\overline{C}_{1,4}K_{2,4}K_{3,4}+B_{3,4}B_{2,4}B_{1,4}K_{1,4}K_{2,4}K_{3,4}\end{aligned}$$

Using Lemma 17, we have (4.18).

Step 3. In the right hand side of (4.18), if we pick up the terms

$$\begin{aligned}&q^2f^{(N)}\Big\{G^{-1}_{N-3,N-2}G^{-1}_{N-4,N-3}\cdots G^{-1}_{2,3}G^{-1}_{1,2}\\&+G^{-1}_{N-4,N-3}G^{-1}_{N-5,N-4}\cdots G^{-1}_{1,2}S_{N-3,N-2}\\&+G^{-1}_{N-5,N-4}\cdots G^{-1}_{1,2}S_{N-3,N-2}S_{N-4,N-3}\\&+\cdots+S_{N-3,N-2}S_{N-4,N-3}\cdots S_{2,3}S_{1,2}\Big\}\times\\&\times\overline{C}_{1,N-1}F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)\Big|_{z_N=q^{-2}z_{N-1}},\end{aligned}$$

they cancel the first term in (4.16). For example, for $N=4$ we have

$$
\begin{aligned}
&(1-q^2)z_3\Big\{f^{(2)}(S_{1,2}+G_{1,2}^{-1})F_{\varepsilon_1\varepsilon_2}(z_2,p^{-1}z_1)(z_2-q^2z_3)\Big\}\\
&+q^2\Big\{f^{(4)}(G_{1,2}^{-1}+S_{1,2})\overline{C}_{1,3}F_{\varepsilon_1\varepsilon_2c^{(1)}}(z_2,z_3,z_4,p^{-1}z_1)\Big\}\Big|_{z_4=q^{-2}z_3}\\
&\sim 0.
\end{aligned}
$$

Lemma 18 $(G_{j,j+1}^{-1}+C_{j+1,N-1})C_{j,N-1}=C_{j+1,N-1}G_{j,j+1}$.

Step 4. Using Lemma 18, we can show the cancellation of the terms obtained by setting formally $G_{1,2}^{-1}=S_{N-3,N-2}=0$ in (4.18). For example, for $N=4$, we have $(C_{2,3}-C_{2,3})F_{\varepsilon_1\varepsilon_2c^{(1)}}(z_2,z_3,z_4,p^{-1}z_1)\Big|_{z_4=q^{-2}z_3}=0$. For $N=5$, we have

$$
\begin{aligned}
&\Big\{(G_{2,3}^{-1}+C_{3,4})C_{2,4}-C_{2,4}C_{3,4}-C_{3,4}(S_{1,2}-C_{2,4})\Big\}\times\\
&\times F_{\varepsilon_1\varepsilon_2\varepsilon_3c^{(1)}}(z_2,z_3,z_4,z_5,p^{-1}z_1)\Big|_{z_5=q^{-2}z_4}\\
&\sim\Big\{C_{3,4}G_{2,3}-C_{2,4}C_{3,4}-C_{3,4}(G_{2,3}-C_{2,4})\Big\}\times\\
&\times F_{\varepsilon_1\varepsilon_2\varepsilon_3c^{(1)}}(z_2,z_3,z_4,z_5,p^{-1}z_1)\Big|_{z_5=q^{-2}z_4}\\
&=0.
\end{aligned}
$$

Here we used $S_{1,2}F_{\varepsilon_1\varepsilon_2\varepsilon_3c^{(1)}}(z_2,z_3,z_4,z_5,p^{-1}z_1)\sim G_{2,3}F_{\varepsilon_1\varepsilon_2\varepsilon_3c^{(1)}}(z_2,z_3,z_4,z_5,p^{-1}z_1)$.

Step 5. The remaining terms on the right hand side of (4.18) are grouped into three categories: the terms containing (i) $G_{1,2}^{-1}$ but not $S_{N-3,N-2}$, (ii) $S_{N-3,N-2}$ but not $G_{1,2}^{-1}$, (iii) both $G_{1,2}^{-1}$ and $S_{N-3,N-2}$.

The terms in (i), (ii), (iii) do not contain $K_{j,N-1}$, $K_{j,N}$ $(1\le j\le N-2)$. Therefore, if we apply these terms to $F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)$ in (4.16), the positions of z_{N-1} and z_N in $F_{\varepsilon_1\cdots\varepsilon_{N-2}c^{(1)}}(z_2,\cdots,z_N,p^{-1}z_1)$ do not change. Therefore, using $\overline{C}_{1,N-1}=\dfrac{q^{-1}(1-q^2)z_{N-1}}{z_{N-1}-z_1}$, the relation (4.16) (to be proved) reduces to the form

$$
\begin{aligned}
&\sum_{j=1}^{N-2}f^{(N-2)}S_{N-3,N-2}\cdots S_{j,j+1}G_{j-1,j}^{-1}\cdots G_{1,2}^{-1}\Delta F\\
&-f^{(N-2)}\Big\{(G_{N-3,N-2}^{-1}+C_{N-2,N-1})\cdots(G_{1,2}^{-1}+C_{2,N-1})\\
&+(G_{N-4,N-3}^{-1}+C_{N-3,N-1})\cdots(S_{N-3,N-2}-C_{N-2,N-1})\\
&+\cdots+(S_{N-3,N-2}-C_{N-2,N-1})\cdots(S_{1,2}-C_{2,N-1})\Big\}\Delta F\\
&\sim 0,
\end{aligned}
\tag{4.19}
$$

where we used the abbreviation

$$
\begin{aligned}
\Delta&=\prod_{k=2}^{N-2}(z_k-q^2z_{N-1}),\\
F&=F_{\varepsilon_1\cdots\varepsilon_{N-2}}(z_2,\cdots,z_{N-2},p^{-1}z_1).
\end{aligned}
$$

Let us write $z_{1'} = p^{-1}z_1$ and

$$G_{N-2,1'} = \frac{q^{-1}z_{N-2} - qz_{1'}}{z_{N-2} - z_{1'}} K_{N-2,1} + \frac{(q-q^{-1})z_{N-2}}{z_{N-2} - z_{1'}}. \tag{4.20}$$

Then, the relation (4.19) reads

$$\begin{aligned}
&\sum_{j=1}^{N-2} f^{(N-2)} S_{N-3,N-2} \cdots S_{j,j+1} G_{j-1,j}^{-1} \cdots G_{1,2}^{-1} \Delta F \\
&-f^{(N-2)} \Big\{ (G_{N-3,N-2}^{-1} + C_{N-2,N-1}) \cdots (G_{1,2}^{-1} + C_{2,N-1}) \\
&+(G_{N-4,N-3}^{-1} + C_{N-3,N-2}) \cdots (G_{N-2,1'} - C_{N-2,N-1}) \\
&+ \cdots + (G_{N-2,1'} - C_{N-2,N-1}) \cdots (G_{2,3} - C_{2,N-1}) \Big\} \Delta F \\
&\sim 0.
\end{aligned}$$

Let us call the terms in the first sum in the left hand side of this relation "the wanted terms" or for short WT. The second sum contains WT. The proof is over if we show the terms in the second sum other than WT (we call them "the unwanted terms" or for short UWT), vanish. We will show this statement by induction on N. For $N = 2$ and $N = 3$, the statement is trivial. For $N = 4$, the statement holds because $C_{2,3} - C_{2,3} = 0$. In general, we show the terms (i), (ii),(iii) other than WT separately cancel. For example, for $N = 5$ we have

$$(G_{2,3}^{-1} + C_{3,4})(G_{1,2}^{-1} + C_{2,4}) + (G_{1,2}^{-1} + C_{2,4})(G_{3,1'} - C_{3,4}) + (G_{3,1'} - C_{3,4})(G_{2,3} - C_{2,4}).$$

UWT in (iii):0

UWT in (i)+(iii):

$$\begin{aligned}
&\Big\{ (G_{2,3}^{-1} + C_{3,4}) G_{1,2}^{-1} + G_{1,2}^{-1}(G_{3,1'} - C_{3,4}) \Big\} - (G_{2,3}^{-1}G_{1,2}^{-1} + G_{1,2}^{-1}G_{3,1'}) \\
&= (C_{3,4} - C_{3,4}) G_{1,2}^{-1} = 0.
\end{aligned}$$

UWT in (ii)+(iii):

$$\begin{aligned}
&\Big\{ (G_{1,2}^{-1} + C_{2,4}) G_{3,1'} + G_{3,1'}(G_{2,3} - C_{2,4}) \Big\} - (G_{1,2}^{-1}G_{3,1'} + G_{3,1'}G_{2,3}^{-1}) \\
&= (C_{2,4} - C_{2,4}) G_{3,1'} = 0.
\end{aligned}$$

For $N = 6$ we have

$$\begin{aligned}
&(G_{3,4}^{-1} + C_{4,5})(G_{2,3}^{-1} + C_{3,5})(G_{1,2}^{-1} + C_{2,5}) \\
&+(G_{2,3}^{-1} + C_{3,5})(G_{1,2}^{-1} + C_{2,5})(G_{4,1'} - C_{4,5}) \\
&+(G_{1,2}^{-1} + C_{2,5})(G_{4,1'} - C_{4,5})(G_{3,4} - C_{3,5}) \\
&+(G_{4,1'} - C_{4,5})(G_{3,4} - C_{3,5})(G_{2,3} - C_{2,5}).
\end{aligned}$$

The cancellation of the terms in (iii) reduces to the case $N = 4$.

The terms in (i)+(iii) are

$$
\begin{aligned}
&(G_{3,4}^{-1}+C_{4,5})(G_{2,3}^{-1}+C_{3,5})G_{1,2}^{-1}\\
&+(G_{2,3}^{-1}+C_{3,5})G_{1,2}^{-1}(G_{4,1'}-C_{4,5})\\
&+G_{1,2}^{-1}(G_{4,1'}-C_{4,5})(G_{3,4}-C_{3,5}).
\end{aligned}
$$

Except for the terms in (iii), which are already done, we can commute $G_{1,2}^{-1}$ with the terms located on the right of $G_{1,2}^{-1}$. By moving $G_{1,2}^{-1}$ to the right end, the statement reduces to the cancellation of UWT in

$$(G_{3,4}^{-1}+C_{4,5})(G_{2,3}^{-1}+C_{3,5})+(G_{2,3}^{-1}+C_{3,5})(G_{4,1'}-C_{4,5})+(G_{4,1'}-C_{4,5})(G_{3,4}-C_{3,5}).$$

This is nothing but the case $N=5$. We omit the further details.

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6. References

1. A. Berkovich, *Fermionic counting of RSOS-states and Virasoro character formulas for the unitary minimal series $M(\nu,\nu+1)$. Exact results*, Nucl. Phys. **B431** (1994), 315.
2. A. Berkovich and B. M. McCoy, *Continued fractions and fermionic representations for characters of $M(p,p')$ minimal models*, 1994, Preprint BONN-HE-94-28, ITPSB 94-060 hep-th/9412030.
3. D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, *Yang-Baxter equation in long-range interacting systems*, J. Phys. A **26** (1993), 5219–5236.
4. D. Bernard, V. Pasquier, and D. Serban, *Spinons in conformal field theory*, Nucl. Phys. **B428** (1994), 612.
5. P. Bouwknegt, A. W. W. Ludwig, and K. Schoutens, *Spinon bases for higher level SU(2) WZW models*, 1994, Preprint USC-94/20, UCSB-TH-94, PUPT-1522, hep-th/9412108.

6. ______, *Spinon bases, Yangian symmetry and fermion representations of Virasoro characters in conformal field theory*, Phys. Lett. **338B** (1994), 448.
7. ______, *Spinon basis for $(\widehat{sl(2)})_k$ integrable highest weight modules*, preprint USC 94/22, PUPT-1537, 1994, to appear in the proceedings of 'Statistical Mechanics and Quantum Field Theory,' USC, May 16-21, 1994.
8. F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971), 419–439.
9. V. Chari and A. Pressley, *Yangians and R-matrices*, L'Enseignement Mathématique **36** (1990), 267–302.
10. ______, *Quantum affine algebras and affine Hecke algebras*, preprint, UCR and KCL, August 1993, q-alg/9501003.
11. I. V. Cherednik, *A new interpretation of Gelfand-Tzetlin bases*, Duke. Math. J. **54** (1987), 563–577.
12. B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Comm. Math. Phys. **151** (1993), 89.
13. V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Doklady **32** (1985), 254–258.
14. ______, *Degenerate affine Hecke algebra and Yangians*, Funct. Anal. Appl. **20** (1986), 62–64.
15. ______, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Doklady **36** (1988), 212.
16. L. D. Faddeev and L. A. Takhtadzhan, *What is the spin of a spin wave?*, Phys. Lett. **85A** (1981), 375–377.
17. ______, *Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model*, J. Soviet Math. **24** (1984), 241–267.
18. B. L. Feigin, T. Nakanishi, and H. Ooguri, *The annihilating ideals of minimal models*, Proceedings of the RIMS Project 1991, Infinite Analysis (A. Tsuchiya, T. Eguchi, and M. Jimbo, eds.), Advanced Series in Mathematical Physics, vol. 16A, World Scientific, Singapore, 1992, pp. 217–238.
19. O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, and H. Yan, *Notes on highest weight modules of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$*, 1994, RIMS preprint RIMS-979 (hep-th/9405058), to appear in Proceedings 'Quantum Field Theory, Integrable Models and Beyond', Eds. T. Inami and R. Sasaki, Suppl. Prog. Theor. Phys.
20. O. Foda and Y-H. Quano, *Polynomial identities of the Rogers Ramanujan type*, 1994, Preprint, hep-th/9407191.
21. I. B. Frenkel and N. Yu Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), 1–60.
22. F. D. M. Haldane, *Exact Jastrow-Gutzwiller resonating-valence-bond ground state of the spin-$1/2$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange*, Phys. Rev. Lett. **60** (1988), 635–638.

23. F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, *Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory*, Phys. Rev. Lett **69** (1992), 2021–2025.
24. M. Jimbo and T. Miwa, *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics vol. **85**, AMS, 1994.
25. R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, *Fermionic quasiparticle representations for characters of* $(G^{(1)})_1 \times (G^{(1)})_1/(G^{(1)})_2$, Phys. Lett. **304** (1993), 263–270.
26. R. Kedem and B. M. McCoy, *Construction of modular branching functions from Bethe's equations in the 3-state Potts chain*, J. Stat. Phys. **71** (1993), 865.
27. A. Kuniba, T. Nakanishi, and J. Suzuki, *Characters in conformal field theories from thermodynamic Bethe ansatz*, Mod. Phys. Lett **A8** (1993), 1649.
28. J. Lepowsky and M. Primc, *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, Contemporary Mathematics **46** AMS, Providence 1985.
29. E. Melzer, *The many faces of a character*, Lett. Math. Phys. **31** (1994), 233.
30. W. Nahm, A. Recknagel, and M. Terhoeven, *Dilogarithm identities in conformal field theory*, Mod. Phys. Lett. **A8** (1993), 1835–1848.
31. A. Nakayashiki and Y. Yamada, *Crystalizing the spinon basis*, 1995, Preprint, hep-th/9504052.
32. B. S. Shastry, *Exact solution of an* $s = 1/2$ *Heisenberg antiferromagnetic chain with long-range interactions*, Phys. Rev. Lett. **60** (1988), 639–642.
33. B. Sutherland, *Exact results for a quantum many–body problem in one dimension. II*, Phys. Rev. A **5** (1972), 1372.
34. S. Ole Warnaar, *Fermionic solution of the Andrews-Baxter-Forrester model: unification of TBA and CTM methods*, 1995, Preprint, hep-th/9501134.